Name ________________________________ Date _________

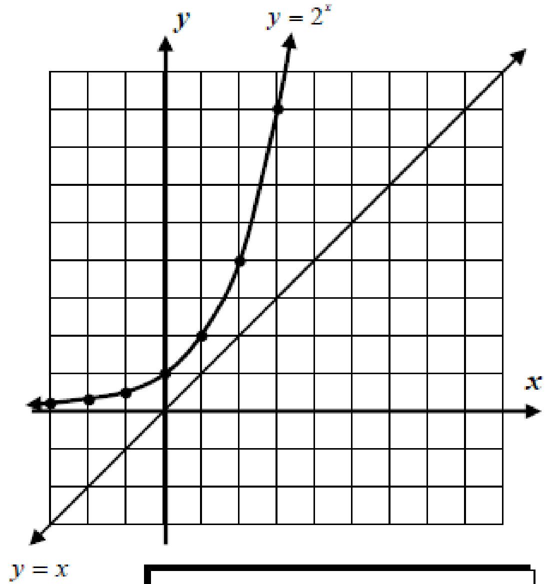
Algebra 2H Unit 10: Logarithmic Functions Notes

Introduction to Logarithms and Their Graphs

Exponential functions are of such importance to mathematics that their inverses, functions that "reverse" their action, are important themselves. These functions, known as ***logarithms****, will be introduced in this lesson.*

Example 1 The function $f(x) = 2^x$ is shown graphed on the axes below along with its table of values.

x	-3	-2	-1	0	1	2	3
$f(x)=2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



Notice that, as always, the graphs of $f(x)$ and $f^{-1}(x)$ are symmetric across $y = x$

(a) Is the function one-to-one? Explain.

(b) Based on your answer from part a, what must be true about the inverse of this function?

(c) Create a table of values below for the inverse of $f(x) = 2^x$ **and** plot this graph on the axes given. *HINT: Switch the x and y.*

x							
$f^{-1}(x)$							

(d) What would be the first step to find an equation for this inverse algebraically? Write this step down and then stop.

Defining Logarithmic Functions – The function $y = log_b\, x$ is the name we give the inverse of $y = b^x$. For example, $y = log_2\, x$ is the inverse of $y = 2^x$. Based on Example 1d, we can write an **equivalent exponential equation** for each logarithm as follows:

$y = log_b\, x$ is the same as $b^y = x$
(*b* is the base, *y* is the exponent, *x* is the answer)

Based on this, we see that a logarithm gives as its output (*y*-value) the exponent we must raise *b* to in order to produce its input (*x*-value).

Example 2 Evaluate the following logarithms. If needed, write an equivalent exponential equation. Do as many as possible without the use of your calculator, then use your calculator to verify. The first one has been done for you.

(a) $log_2\, 8 = \boxed{\mathbf{3}}$ (b) $log_4\, 16 =$ (c) $log_6\, 1 =$

Because: $2^{\square} = 8$ So, $2^3 = 8$

(d) $log_6 \frac{1}{6} =$ (e) $log_6 \frac{1}{36} =$ (f) $log_6 \sqrt{6} =$

Example 3 The log form of $y = a^x$ is

(1) $y = log_a\, x$ (3) $a = log_x\, y$

(2) $x = log_a\, y$ (4) $x = log_y\, a$

Example 4 Which of the following is equivalent to $y = log_4\, x$?

(1) $y = x^4$ (3) $x = 4^y$

(2) $x = y^4$ (4) $y = 4^x$

Example 5 Which of the following represents the inverse of $y = log_4\, x$?

(1) $y = x^4$ (3) $x = 4^y$

(2) $x = y^4$ (4) $y = 4^x$

Calculator Use and Logarithms – Most non-graphing calculators only have two logarithms that they can evaluate directly. One of them, $log_{10}\, x$, is so common that it is actually called the **common log** and typically is written without the base 10.

$log\, x = log_{10}\, x$ (The Common Log)

Example 6 Evaluate each of the following **without** using your calculator.

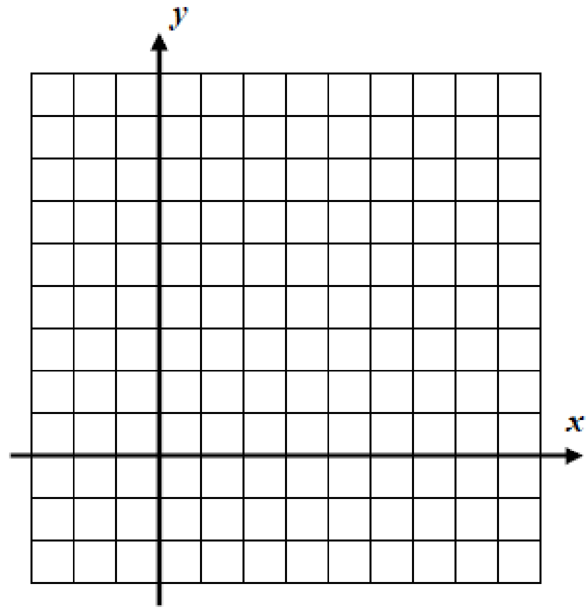
(a) $log\,100 =$ (b) $log\,\frac{1}{1000} =$ (c) $log\,10 =$

Example 7 Consider the logarithmic function $y = log_3 x$ and its inverse $y = 3^x$.

(a) Construct a table of values for $y = 3^x$ and then use this to construct a table of values for the function $y = log_3 x$. Graph and label both functions.

x	-2	-1	0	1	2
$y = 3^x$					

x					
$y = \log_3 x$					



(b) State the domain and range of $y = 3^x$ and $y = log_3 x$. State your answers in set-builder notation.

$y = 3^x$

Domain:

Range:

$y = log_3 x$

Domain:

Range:

(c) Identify the type of asymptote and its equation for $y = 3^x$ and $y = log_3 x$.

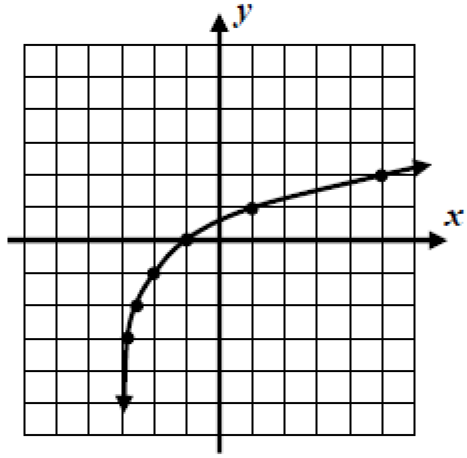
$y = 3^x$

$y = log_3 x$

Notice how switching the *x* and *y* to create the inverse also switches the domain and range and the asymptotes.

Example 8 Which of the following equations describes the graph shown below? Show or explain how you made your choice.

(1) $y = log_2(x + 3) - 1$
(2) $y = log_2(x - 3) - 1$
(3) $y = log_2(x + 3) + 1$
(4) $y = log_2(x - 3) + 1$



Example 9 The fact that finding the logarithm of a non-positive number (negative or zero) is not possible in the real number system allows us to find the domains of a variety of logarithmic functions.

(a) Determine the domain of the function graphed in Example 8.

(b) Determine the domain of the function $y = log_2(3x - 4)$. *HINT: Set* $3x - 4 > 0$ *and solve. This inequality will guarantee the domain is not 0 and stays positive.*

Logarithm Laws

Logarithms have properties, just as exponents do, that are important to learn because they allow us to solve a variety of problems where logarithms are involved. Keep in mind that since logarithms give exponents, the laws that govern them should be similar to those that govern exponents. Below is a summary of these laws.

LAW	EXPONENT VERSION	LOGARITHM VERSION
Product	$b^x \cdot b^y = b^{x+y}$	$\log_b(x \cdot y) = \log_b x + \log_b y$
Quotient	$\dfrac{b^x}{b^y} = b^{x-y}$	$\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
Power	$\left(b^x\right)^y = b^{x \cdot y}$	$\log_b\left(x^y\right) = y \cdot \log_b x$

Example 10 Which of the following is equal to $log_3(9x)$?

(1) $log_3 2 + log_3 x$ (3) $2 + log_3 x$
(2) $2\, log_3 x$ (4) $log_3 2 + x$

Example 11 The expression $log\left(\frac{x^2}{1000}\right)$ can be written in equivalent form as

(1) $2\, log\, x - 3$ (3) $2\, log\, x - 6$
(2) $log\, 2x - 3$ (4) $log\, 2x - 6$

Example 12 If a = log3 and b = log 2 then which of the following correctly expresses the value of log12 in terms of a and b?

(1) $a^2 + b$ (3) $2a + b$
(2) $a + b^2$ (4) $a + 2b$

Example 13 Which of the following is equivalent to $log_2\left(\frac{\sqrt{x}}{y^5}\right)$?

(1) $\sqrt{log_2 x} - 5\, log_2 y$ (3) $\frac{1}{2} log_2 x - 5\, log_2 y$
(2) $2\, log_2 x + 5\, log_2 y$ (4) $2\, log_2 x - 5\, log_2 y$

Example 14 The value of $log_3\left(\frac{\sqrt{5}}{27}\right)$ is equal to

(1) $\frac{log_3 5-6}{2}$ (3) $2\,log_3 5+3$
(2) $\frac{log_3 5-3}{2}$ (4) $2\,log_3 5-3$

Example 15 If $f(x)=log(x)$ and $g(x)=100x^3$ then $f(g(x))=$

(1) $100\,log\,x$ (3) $6+log\,x$
(2) $300\,log\,x$ (4) $2+3\,log\,x$

Example 16 The logarithmic expression $log_2\sqrt{32x^7}$ can be written as

(1) $\sqrt{log_2 3\,5x}$ (3) $\sqrt{5+7\,log_2 x}$

(2) $\frac{5+7\,log_2 x}{2}$ (4) $\frac{35+log_2 x}{2}$

Example 17 If $log\,7=k$ then $log(4900)$ can be written in terms of k as

(1) $2(k+1)$ (3) $2(k-3)$
(2) $2k-1$ (4) $2k+1$

Solving Exponential Equations Using Logarithms

In the previous unit, we used the **Method of Common Bases** to solve exponential equations. This technique is quite limited, however, because it requires the two sides of the equation to be expressed using the same base. A more general method utilizes our calculators and the third logarithm law:

<u>The Third Log Law (The Power Law)</u>

$$log_b(a^x) = x\, log_b\, a$$

Example 18 Solve $4^x = 8$ using (a) common bases and (b) the logarithm law shown above.

(a) Method of Common Bases: $4^x = 8$

(b) Logarithm Approach: $4^x = 8$

Step 1: Take the common log of both sides.
Step 2: Use the Power Log Law to simplify the expression
Step 3: Solve for *x*.

The beauty of this logarithm law is that it removes the variable from the exponent. This law, in combination with the logarithm base 10, the ***common log****, allows us to solve almost any exponential equation using calculator technology.*

Example 19 **<u>Using logs</u>**, solve each of the following equations for the value of *x*. Round your answers to the nearest *hundredth*.

These equations can become more complicated, but each and every time we will use the power logarithm law to transform an exponential equation into one that is more familiar (linear only for now).

(a) $5^x = 18$ (b) $4^x = 100$ (c) $2^x = 1560$

(d) $6^{x+3} = 50$ (e) $(1.03)^{\frac{1}{2}x-5} = 2$

Many modern calculators (including the nSpires) can find a logarithm of any base. Some still only have the common log (base 10) and another that we will soon see. But, we can still express our answers in terms of logarithms and evaluate them.

Example 20 Find the solution to each of the following exponential equations in terms of a logarithm and then evaluate your expression to the nearest hundredth. *HINT: Isolate the exponential expression first and then rewrite your exponential equation into log form.*
Remember: $y = b^x$ *is the same as* $x = log_b\, y$

(a) $4(2)^x - 3 = 17$

(b) $17(5)^{\frac{x}{3}} = 4$

Now that we are familiar with this method, we can revisit some of our exponential models from the last unit. Recall that for an exponential function that is increasing or decreasing:

If quantity Q is known to increase/decrease by a fixed percentage p, in decimal form, then Q can be modeled by

$$Q(t) = Q_0(1 \pm p)^t$$

where Q_0 represents the amount of Q present at t = 0 (initial amount or value) and t represents time.

Example 21 A biologist is modeling the population of bats on a tropical island. When he first starts observing them, there are 104 bats. The biologist believes that the bat population is growing at a rate of 3% per year.

(a) Write an equation for the number of bats, $B(t)$, as a function of the number of years, t, since the biologist started observing them.

(b) Using your equation from part (a), **algebraically (with logs)** determine the number of years it will take for the bat population to reach 200. Round your answer to the nearest year.

Example 22 A stock has been declining in price at a steady pace of 5% per week. If the stock started at a price of $22.50 per share, determine **algebraically** the number of weeks it will take for the price to reach $10.00. Round your answer to the nearest week.

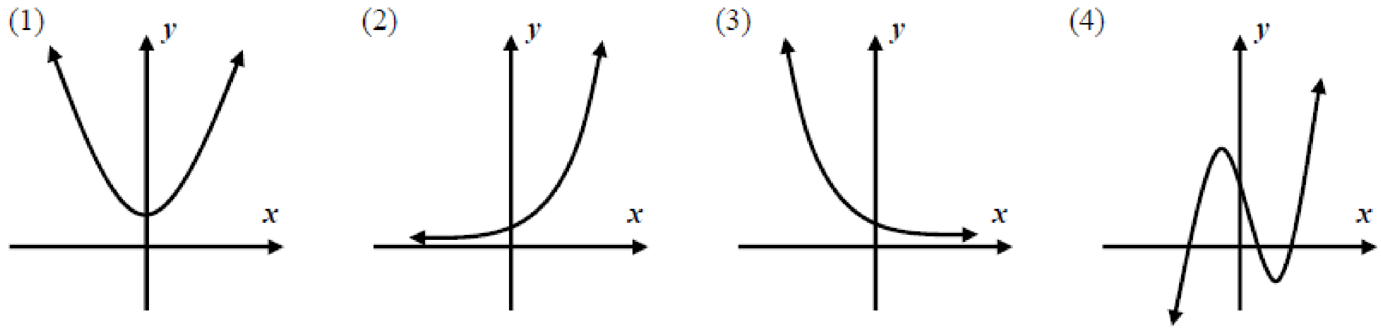
The Number *e* and the Natural Logarithm

There are many numbers in mathematics that are more important than others because they find so many uses in either mathematics or science. Good examples of important numbers are 0, 1, *i*, and π. In this lesson you will be introduced to an important number given the letter **e** for its "inventor" Leonhard Euler (1707-1783). This number plays a crucial role in Calculus and more generally in modeling exponential phenomena.

THE NUMBER e

1. Like π, e is irrational.
2. $e \approx 2.72$
3. Used in Exponential Modeling

Example 23 Which of the graphs below shows $y = e^x$? Explain your choice. Check on your calculator.



Explanation:

Since e is essentially like Pi (has a numerical equivalent), then it will act like a base > 1. Just like last unit.

Very often e is involved in exponential modeling of both increasing and decreasing quantities.

Example 24 A population of llamas on a tropical island can be modeled by the equation $P(t) = 500e^{0.035t}$, where *t* represents the number of years since the llamas were first introduced to the island.

(a) How many llamas were initially introduced at *t* = 0 ? Show the calculation that leads to your answer.

$P(t) = 500e^{0.035(0)} = 500\ (e^0) = 500\ (1) = 500$ ← our starting # of llamas

(b) **<u>Algebraically</u>** determine the number of years for the population to reach 600. Round your answer to the nearest *tenth* of a year.

$500e^{0.035(t)} = 600$	Now try to isolate t; divide both sides by 500
$e^{0.035(t)} = 6/5$	Take the natural log (ln) of both sides
$\ln\ (e^{0.035(t)}) = \ln\ (6/5)$	Use the power law to bring exponent down
$0.035t\ (\ln e) = \ln\ (6/5)$	Since $\ln e = \log_e e$, then $\ln e = 1$
$0.035t = \ln\ (6/5)$	Now use a calc and divide both sides by 0.035; t = 5.2 yrs

Because of the importance of $y = e^x$ its **inverse**, known as the **natural logarithm**, is also important.

THE NATURAL LOGARITHM

The inverse of $y = e^x$: $\quad y = \ln x \quad (y = \log_e x)$

*The natural logarithm, like all logarithms, gives an exponent as its output. In fact, it gives the power that we must raise **e** to in order to get the input.*

Example 25 ***Without*** the use of your calculator, determine the values of each of the following. Afterwards use your calculator to check.

(a) $ln(e)$ = 1

(c) $ln(e^5)$ = 5 (ln e) = 5 (1) = 5

(b) $ln(1)$ = 0

(d) $ln(\sqrt{e})$ = 1/2

The natural logarithm follows the three basic logarithm laws that all logarithms follow.

$$ln(xy) = ln\,x + ln\,y$$
$$ln\left(\frac{x}{y}\right) = ln\,x - ln\,y$$
$$ln(x^y) = y\,ln\,x$$

Example 26 Which of the following is equivalent to $ln\left(\frac{x^3}{e^2}\right)$? = ln ($x^3$) – ln ($e^2$)

(1) $ln\,x + 6$

(2) $3\,ln\,x - 2$

(3) $3\,ln\,x - 6$

(4) $ln\,x - 9$

= 3 (ln x) – 2 (ln e)

= 3 ln (x) – 2 ← since ln e = 1

Example 27 A hot liquid is cooling in a room whose temperature is constant. Its temperature can be modeled using the exponential function shown below. The temperature, *T*, is in degrees Fahrenheit and is a function of the number of minutes, *m*, it has been cooling.

$$T(m) = 101e^{-0.03m} + 67$$

(a) What was the initial temperature of the water at $m = 0$. Do without using your calculator.

$T(0) = 101e^{-0.03(0)} + 67 = 101\,e^{0} + 67 = 101 + 67 = 168$

(b) How do you interpret the statement that $T(60) = 83.7$?

After 60 minutes, the temp of the liquid has dropped to 83.7 degrees

(c) Using the natural logarithm, determine **algebraically** when the temperature of the liquid will reach 100 ∘F. Show the steps in your solution. Round to the nearest tenth of a minute. *HINT: Solve the equation as we have been but use the natural log instead of the common log*.

$101\,e^{-0.03t} + 67 = 100$	Create an equation to solve for t
$101\,e^{-0.03t} = 33$	Subtract 67 from both sides
$e^{-0.03t} = 33/101$	Divide by 101 (leave the fraction)
$\ln(e^{-0.03t}) = \ln(33/101)$	Take natural log of both sides
$-0.03t(\ln e) = \ln(33/101)$	Use power rule to bring exponent down
$-0.03t = \ln(33/101)$	Drop the ln (e) since it's equal to 1
t = 37 minutes	Use calc to solve for t

Please realize that I am doing all of this with a scientific calculator (actually the one on my iPhone), so not having a graphing calculator should not be a deterrent to working.

(d) On average, how many degrees are lost per minute over the interval $10 \le m \le 30$? Round to the nearest tenth of a degree. *HINT: You are finding the average rate of change – slope*.

Use your calculator to plug in 10 and 30 into your original equation. Then use average rate of change formula to solve.

T (10) = 141.82 degrees
T (30) = 108.06 degrees

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{108.06 - 141.82}{30 - 10} = \frac{-33.76}{20} = 1.7 \text{ degrees/minute}$$

Exact Answers

There are times when an exact answer is warranted with logarithms. In this case, we will need to manipulate the equation to solve for the specified variable. Let's try one.

Example 28 $$10 = \frac{40}{1+7e^{-0.25t}}$$

THIS IS ONE OF OUR EXTRA TOPICS THAT WE WILL SKIP SINCE

TIME WILL BE OF THE ESSENCE IF WE RETURN.

Compound Interest

In the worlds of investment and debt, interest is added onto a principal in what is known as ***compound interest****. The percent rate is typically given on a yearly basis but could be applied more than once a year. This is known as the* ***compounding frequency****. Let's look at a typical problem to understand how the compounding frequency changes how interest is applied.*

Example 29 A person invests \$500 in an account that earns a **nominal yearly interest rate** of 4%.

(a) How much would this investment be worth in 10 years if the **compounding frequency** was once per year? Show the calculation you use.

$V(t) = 500\,(1.04)^{10} = 740.12$

(b) If, on the other hand, the interest was applied four times per year (known as quarterly compounding), why would it not make sense to multiply by 1.04 each quarter?

This would give us 4% per quarter, which isn't realistic!

(c) If you were told that an investment earned 4% per year, how much would you assume was earned per quarter? Why?

We would assume 1% applied four times over the year

(d) Using your answer from part (c), calculate how much the investment would be worth after 10 years of quarterly compounding? Show your calculation.

$V(t) = 500\,(1.01)^{40}$ This is us applying 1% forty times (4x per year)

$V(t) = 744.43$ It is slightly higher since we applied the interest more often

So, the pattern is straightforward. For a **shorter compounding period**, we get to **apply the interest more often**, but at a **lower rate**. Let's formalize this pattern into a classic formula from economics for the amount, *A*, the investment is worth after *t*-years:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

P = amount initially invested

r = nominal yearly rate

n = number of compounds per year

Example 30 The rate in the previous formula was referred to as **nominal** (**in name only)**. It's known as this, because you effectively earn more than this rate if the compounding period is more than once per year. Because of this, bankers refer to the **effective rate**, or the rate you would receive if compounded just once per year. Let's investigate this.

An investment with a nominal rate of 5% is compounded at different frequencies. Give the **effective** yearly rate, accurate to two decimal places, for each of the following compounding frequencies. Show your calculation.

(a) Quarterly

$(1 + \frac{.05}{4})^4 = 1.0509$

So about 5.10%

(b) Monthly

$(1 + \frac{.05}{12})^{12} = 1.0511$

So about 5.11%

(c) Daily

$(1 + \frac{.05}{365})^{365} = 1.0513$

So about 5.13%

Notice how each time we changed the compounding, it raised the overall percentage slightly? This is how interest works. The more time it is compounded, there is a slight increase in the overall rate.

Example 31 How much would $1000 invested at a nominal 2% yearly rate, compounded monthly, be worth in 20 years? Show the calculations that lead to your answer.

(1) $1485.95

(3) $1033.87

$1000\,(1 + \frac{.02}{12})^{12*20} = 1491.3280$

(2) $1491.33

(4) $1045.32

We could compound at smaller and smaller frequency intervals, eventually compounding all moments of time. In our formula, we would be letting n approach infinity. This gives rise to **continuous compounding** and the use of the natural base **e** in the famous **continuous compound interest formula:**

For an initial principal, P, compounded continuously at a nominal yearly rate of r, the investment would be worth an amount A given by:

$$A(t) = Pe^{rt}$$

Example 32 A person invests \$350 in a bank account that promises a nominal rate of 2% continuously compounded.

(a) Write an equation for the amount this investment would be worth after t-years.

$A(t) = 350\, e^{.02t}$

(b) How much would the investment be worth after 20 years?

$A(20) = 350\, (e^{.02*20}) = 522.14$

(c) **Algebraically** determine the time it will take for the investment to reach \$400. Round to the nearest tenth of a year.

$350\, e^{.02t} = 400$	Set original equation equal to the target amount of \$400
$e^{.02t} = 8/7$	Divide both sides by 350 … reduces to 8/7
$\ln (e^{.02t}) = \ln (8/7)$	Take the natural log of both sides (since we're using e)
$.02t (\ln e) = \ln (8/7)$	Use power rule to bring down exponent
$.02t = \ln (8/7)$	Cancel ln e since it's equal to zero; solve for t
$t = 6.7$ years	

(d) What is the effective annual rate for this investment? Round to the nearest hundredth of a percent.

$e^{.02} = 1.0202 = 2.02\%$	This is the effective annual rate (t = 1)